A Language of Case Differences

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Abstract. This paper contributes to a line of research that consists in applying qualitative reasoning techniques to the formalization of the case-based inference, and in particular, to its adaptation phase. The importance of capturing case differences has long been acknowledged in adaptation research, but research is still needed to properly represent and reason upon case differences. Assuming that case differences can be expressed as a set of feature differences, we show that Category Theory can be used as a mathematical framework to design a qualitative language in which both case differences, similarity paths and adaptation rules can be represented and reasoned upon symbolically.

Keywords: case differences · qualitative modeling · similarity path · adaptation

1 Introduction

Qualitative modeling provides formalisms that focus on how people represent themselves and reason about dynamical systems. Qualitative representations partition continuous quantities, and turn them into entities that can be reasoned upon symbolically [8]. The case-based inference aims at finding a complete description of a target problem by transferring information from a set of past problem-solving episodes, called cases, that are indexed in memory. Adaptation is the part of this process that aims at modifying a retrieved case when it can not be reused as it is in the new situation. Previous work on applying qualitative modeling techniques to adaptation includes a qualitative representation of relationships between quantities (called *variations* in [3]), such that co-occurrences of variations (called *co-variations* in [4]) can be interpreted as qualitative proportionalities. These proportionalities have been shown in [4] to play a great role in different commonsense inferences, and in particular, it has been suggested that adaptation was essentially an “analogical jump” performed on such proportionalities.

In existing formalizations, adaptation is recognized as being part of the case-based reasoning cycle [1], but surprisingly, the adaptation step is not included in the case-based analogical inference [17]. A study of the literature shows the adaptation step is always performed *after* the analogical inference (*i.e.*, retrieval, mapping, and transfer) has taken place, and only aims at modifying its result.

Some adaptation methods such as critique-based adaptation [11], or conservative adaptation [14] are used to resolve inconsistencies in the reused source case, whereas others such as differential adaptation [9], case-based adaptation [6] or adaptation by reformulation [15] modify the reused source case in order to fit the requirements on the target case. One of the reasons why adaptation is left out of the case-base inference is that adaptation essentially consists in reasoning on the *differences* that exist between two cases. While the importance of capturing case differences has long been acknowleged in adaptation research (see for example [13], for a recent review), research is still needed to properly represent and reason on case differences.

Establishing a difference between two states is the result of a comparison process. Comparisons are qualitative judgements that play an important role in similarity assessment and in the analogical inference. According to [12], "a comparison assembles two elements in order to come up with a third term that will tell their relationship". Comparison involves three ideas: the *source* of the comparison, the *target* of the comparison (what the source is compared to), and their relationship. For example, one could compare a sheep (the source) to a goat (the target), on how they forage (their relationship): a sheep would graze, whereas goats are browsers. Comparisons are usually made with respect to a particular *feature* (or property), shared by the objects under comparison, and which can be measured, like the size, the weight, or the type of forage [21]. Some results even suggests that people use aggregated features inferred from the features of individual objects to compare collections of objects [20].

Assuming that case differences can be expressed as a set of differences in feature value, we show that Category Theory can be used as a mathematical framework to design a qualitative language in which both case differences, but also "horizontal" connections of variations (similarity paths), and "vertical" connections (adaptation rules) can be represented and reasoned upon symbolically.

The paper is organized as follows. The next section provides some preliminary definitions. Feature comparisons are modeled in Sec. 3 as labeled arrows, and formalized in Sec. 4 as morphisms of a category. Two constructions are made on such categories: products (Sec. 5), and paths (Sec.6). In Sec.7, comparisons are ordered by generality using a subsumption relation. Finally, Sec.8 concludes the paper.

2 Preliminaries

Category theory is the mathematical study of algebras of functions [2]. A category consists of a set of *objects* and a set of *arrows*. For each arrow f, there are given objects $\mathtt{dom}(f)$ and $\mathtt{cod}(f)$ called the *domain* and the *codomain* of f. We write $f : A \longrightarrow B$ to indicate that $\mathtt{dom}(f) = A$ and $\mathtt{cod}(f) = B$. For two arrows f and g such that $\mathtt{cod}(f) = \mathtt{dom}(g)$, there is a given arrow $g \circ f$ called the *composite* of f and g. For each object A, there is a given arrow $1_A : A \longrightarrow A$ called the *identity arrow* of A. Arrows satisfy the *associativity law*: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$.

Identity arrows verify $f \circ 1_A = 1_B \circ f = f$ for all $f : A \longrightarrow B$. An arrow $f : A \longrightarrow B$ is called an *isomorphism* if there is an arrow $g : B \longrightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. A *groupoid* is a category in which every arrow is an isomorphism. The category **Rel** is the category where objects are sets and arrows are binary relations. The identity arrow on a set A is the identity relation: $1_A = \{(a, a) \in A \times A \mid a \in A\}$. Given $f \subseteq A \times B$ and $g \subseteq B \times C$, the composition $g \circ f$ is defined as: $(a, c) \in g \circ f$ iff $\exists b \in B \mid (a, b) \in f$ and $(b, c) \in g$. Categories are mathematical structures which underlying structure is a quiver, *i.e.*, a directed graph where loops and multiple arrows between two vertices are allowed, on which the definition of a category adds constraints on identity morphisms, associativity, and composition. A *path* in the graph of a category is a sequence $\xrightarrow{c_1}\xrightarrow{c_2} \dots \xrightarrow{c_n}$ of arrows of $\mathbf{C}$ such that for all i, $\mathtt{dom}(\xrightarrow{c_{i+1}}) = \mathtt{cod}(\xrightarrow{c_i})$ A *path category* (or free category) generated by a directed graph is the category where the objects are vertices, and arrows are paths between objects. A *functor* $F : \mathbf{C} \longrightarrow \mathbf{D}$ between two categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping of objects to objects and arrows to arrows that preserves domain and codomains, identities, and composition: $F(f : A \longrightarrow B) = F(f) : F(A) \longrightarrow F(B)$, $F(1_A) = 1_{F(A)}$, and $F(g \circ f) = F(g) \circ F(f)$. The *product* $\mathbf{C} \times \mathbf{D}$ of two categories $\mathbf{C}$ and $\mathbf{D}$ is the category of pairs and arrows. Its objects have the form (C, D), for $C \in \mathbf{C}$ and $D \in \mathbf{D}$, and its arrows have the form $(f, g) : (C, D) \longrightarrow (C', D')$ for $f : C \longrightarrow D \in \mathbf{C}$ and $g : C' \longrightarrow D' \in \mathbf{D}$. Compositions and units are defined componentwise, *i.e.*, $(f', g') \circ (f, g) = (f' \circ f\,,\, g' \circ g)$, and $1_{\mathbf{C} \times \mathbf{D}} = (1_{\mathbf{C}}, 1_{\mathbf{D}})$.

3 Modeling Feature Differences

We are interested in modeling the comparison between two values of a same feature. In the following, the term *feature* denotes either a binary variable (*i.e.*, a variable which takes one of the two values 0 or 1), or a nominal variable (*i.e.*, a variable which takes nominal values, like the color), or a quantity (*i.e.*, a variable which take values on ordinal, interval, or ratio scales [18]). The term *feature space* denotes the set of values taken by a particular feature.

A straightforward way to represent a comparison from a source A to a target B is to trace an arrow from A to B and to label this arrow with a term that represents their relationship. For example, an arrow named $\mathtt{g} \to \mathtt{b}$ can be used to represent the relationship in which the forage differs from **g**(raze) to **b**(rowse) from source to target (Fig. 1). The distinction between the source and the target

$$A \xrightarrow{\mathtt{g} \to \mathtt{b}} B$$

Fig. 1: A comparison of two feature values represented by a labeled arrow.

of a comparison makes the process by essence directional. It can be noted that this remains true even if the underlying relation is symmetrical. To illustrate this, consider the symmetrical binary relation `brother`, which relates two people when they are brothers. For two brothers A and B, both $\texttt{brother}(A, B)$ and $\texttt{brother}(B, A)$ hold (by symmetry), but $A \xrightarrow{\texttt{brother}} B$ and $B \xrightarrow{\texttt{brother}} A$ represent two different comparisons.

When the source and the target of the comparison are values of a same feature space, the comparison relation is transitive: if A can be compared with B and B with C, then A can be compared with C [5]. Besides, the relation is invertible, by which we mean that it is not possible to compare an object A to an object B without also being able to "reverse the viewpoint" and compare B with A with another relationship (possibly the same). For example, if a sheep A can be compared to a goat B with the relationship $\texttt{g} \rightarrow \texttt{b}$ ($\texttt{g}$ stands for **g**raze, and $\texttt{b}$ for **b**rowse), then an inverse relationship $\texttt{b} \rightarrow \texttt{g}$ can be used to compare B to A. It can be noted that feature value comparisons constitute a special case among similarity relationships. In the general case, similarity relationships are neither transitive nor invertible. For example, if Ted went to the same school as John and John went to the same school as Mary, it does not entail that Ted went to the same school as Mary. Comparisons may also not be invertible in simili ("a tree is like a man") or metaphors ("love is a battlefield"): we might say "a man is like a tree", meaning that a man has roots, but not "a tree is like a man" [21].

4 Formalization

Feature spaces can be formalized as categories, which we will call *feature categories*. The objects are the values of the feature space, and arrows represent comparisons between these values. Category Theory seems to be a natural setting to represent such comparisons, since arrows (also called morphisms) are the main "building blocks" of categories as mathematical structures. The categorical notion of composition of arrows corresponds to the transitivity of the comparison relation. Besides, each object of a category must be related to itself by an identity arrow. So representing a feature space as a category requires to distinguish identity arrows from difference arrows. Identity arrows, like $\texttt{d} \rightarrow \texttt{d}$ or $=$, have the same object as origin and destination, and express commonalities. Difference arrows, like $\texttt{d} \rightarrow \texttt{m}$ of $<$, have different origin and destination objects, and express differences. As all arrows are invertible in a feature category, the obtained category is a groupoid.

For example, the category **Bin** (Fig. 2) represents the quantity space of Boolean values, by taking as objects the two Boolean values 1 (True) and 0 (False), and as arrows the possible comparisons between these values. Feature categories may also represent quantity spaces. For example, consider the category $\mathbf{C}_<$, in which objects are elements of $\mathbb{N}$, and there are three arrows $\xrightarrow{=}$, $\xrightarrow{<}$, and $\xrightarrow{>}$. The arrow $\xrightarrow{=}$ is the identity arrow that links every integer $x \in \mathbb{N}$ to itself. The arrow $\xrightarrow{<}$ (resp., $\xrightarrow{>}$) links two integers x and y whenever $x < y$ (resp., $x > y$). Every arrow $\xrightarrow{<}$ is invertible since $y > x$ holds whenever $x < y$. The category **Area** (Fig. 3) represents location areas of apartments. Its objects are

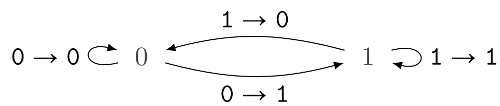

Fig. 2: The example category **Bin**, in which objects represent the Boolean values `0` and `1`, and arrows represent comparisons between these values.

the three nominal values **d**(owntown), **m**(idtown), and **u**(uptown), and its arrows the nine possible comparisons between them.

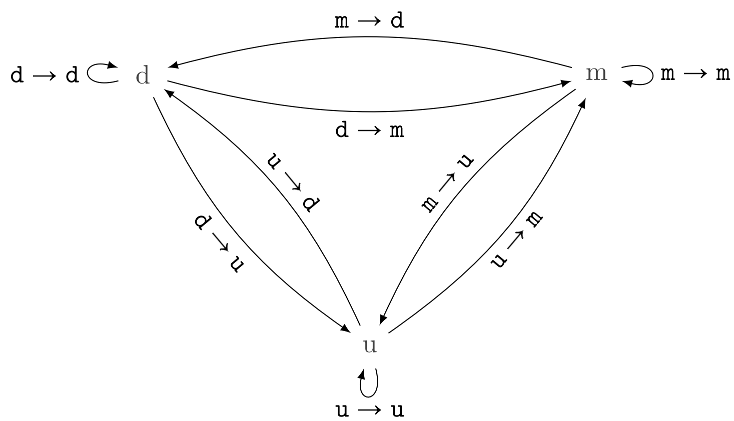

Fig. 3: The example category **Area**, in which objects represent the three areas `d`(*owntown*), `m`(*idtown*), and `u`(*uptown*), and arrows represent comparisons between areas.

4.1 Semantics

Feature categories are interpreted on a set (like a set of patients, of cooking recipes, etc.). Let $\mathcal{X}$ denote such a set. The semantics of a feature category $\mathbf{C}$ on a set $\mathcal{X}$ is given by a functor $.^{\mathcal{I}} : \mathbf{C} \longrightarrow \mathbf{Rel}$, called the *interpretation functor*, which maps each object of the category $\mathbf{C}$ to a subset of $\mathcal{X}$, and arrows to subsequent binary relations. The functor $.^{\mathcal{I}}$ generalizes the notion of binary variation. The definition of a binary variation as proposed in [3] corresponds to the indicator function of $.^{\mathcal{I}}$, when it is restricted to a given arrow of $\mathbf{C}$.

If there exists a *field* function $\varphi : \mathcal{X} \longrightarrow \mathbf{C}$, which maps each element of $\mathcal{X}$ to an object of $\mathbf{C}$, the interpretation functor $.^{\mathcal{I}}$ can be defined to map each object a of $\mathbf{C}$ to its inverse image by φ in $\mathcal{X}$, *i.e.*, to the set of elements of $\mathcal{X}$ which take the value a for the property φ:

$$a^{\mathcal{I}} = \{x \in \mathcal{X} \mid \varphi(x) = a\} \qquad \text{for an object } a \text{ of } \mathbf{C}$$

$$(a \to b)^{\mathcal{I}} = a^{\mathcal{I}} \times b^{\mathcal{I}} \subseteq \mathcal{X} \times \mathcal{X} \qquad \text{for an arrow } a \to b \text{ of } \mathbf{C}$$

For example, let $\mathcal{X}$ be a set of patients, and $\varphi : \mathcal{X} \longrightarrow \mathbf{C}_<$ be a field function that associates to each element of $\mathcal{X}$ an object of the category $\mathbf{C}_<$, representing the age of the patient. The interpretation functor $.^{\mathcal{I}} : \mathbf{C}_< \longrightarrow \mathbf{Rel}$ maps each age value $n \in \mathbb{N}$ to the set of patients having that age, and maps each comparison to the corresponding binary relation. The binary relation $(\xrightarrow{<})^{\mathcal{I}}$ is the set of pairs (a, b) of patients such that b is (strictly) older than a. Likewise, let $\mathcal{X}$ be a set of apartments, and $\varphi : \mathcal{X} \longrightarrow \mathbf{Area}$ be a field function that associates to each element of $\mathcal{X}$ an object of the category **Area**. The interpretation functor $.^{\mathcal{I}} : \mathbf{Area} \longrightarrow \mathbf{Rel}$ maps each nominal value to the set of apartments having the corresponding location area, and maps each comparison to the corresponding binary relation. The binary relation $(\xrightarrow{\mathtt{m}\to\mathtt{d}})^{\mathcal{I}}$ is the set of pairs (a, b) of apartments such that a is located in **m**idtown and b is located in **d**owntown.

5 Representing Differences on Multiple Features

The *product* $\mathbf{C}_1 \times \mathbf{C}_2 \times \ldots \times \mathbf{C}_n$ of n comparison categories $\mathbf{C}_1$, $\mathbf{C}_2$,...,$\mathbf{C}_n$ has as objects the n-tuples $(a_1, a_2, \ldots, a_n)$ where a_i is an object of $\mathbf{C}_i$, and as arrows the n-tuples $(a_1 \to b_1, a_2 \to b_2, \ldots, a_n \to b_n)$, where $a_i \to b_i$ is an arrow of $\mathbf{C}_i$. For example, $(\xrightarrow{\mathtt{m}\to\mathtt{d}}, \xrightarrow{=})$ is an arrow in the product $\mathbf{Area} \times \mathbf{C}_<$, and could be used to represent the comparison between an apartment located in **m**idtown and an apartment located in **d**owntown, both having the same price.

The interpretation functor $.^{\mathcal{I}}$ is extended to products in such a way that an element $x \in \mathcal{X}$ is in the interpretation of the product if it is common to all interpretations of $\mathbf{C}_i$'s:

$$(a_1, a_2, \ldots, a_n)^{\mathcal{I}} = \bigcap_i a_i{}^{\mathcal{I}} \qquad \text{for } n \text{ objects } a_i \text{ of } \mathbf{C}_i$$

$$(a_1 \to b_1, a_2 \to b_2, \ldots, a_n \to b_n)^{\mathcal{I}} = \bigcap_i (a_i \to b_i)^{\mathcal{I}} \qquad \text{for } n \text{ arrows } a_i \to b_i \text{ of } \mathbf{C}_i$$

6 Similarity

6.1 Analogy as Shared Differences

Two pairs are analogous when the same comparison can be made between them. When comparisons represent relations, this idea is consistent with the idea of analogy as a transfer of a relational structure, as outlined by Structure-mapping Theory [10]. For example, in the Andromeda galaxy, the X12 planets resolve around the X12 star, which can be represented as comparisons of the form "A:X12 planet $\xrightarrow{\text{resolve around}}$ X12 star". An analogy can be made between the

Andromeda galaxy and the solar system, by mapping these comparisons with comparisons such as "A:solar system planet $\xrightarrow{\text{resolve around}}$ sun". But the idea of analogy as shared comparisons can be generalized to the comparisons made to establish feature differences, that do *not* represent relations. For example, a same comparison $\texttt{g} \to \texttt{b}$ can be made from a sheep to a goat and from a cow to a moose: cow **g**raze, whereas moose **b**rowse. As a result, a cow is to a sheep what a moose is to a goat.

The same idea can be applied to logical proportions, which can be seen as shared comparisons. For two propositional variables x and y, there are four indicators: $I_1(x, y) = x \wedge y$, $I_2(x, y) = x \wedge \overline{y}$, $I_3(x, y) = \overline{x} \wedge y$, and $I_4(x, y) = \overline{x} \wedge \overline{y}$, and each logical proportion is defined by two distinct equivalences between these indicators [19]. For example, two pairs (x, y) and (z, t) are in *analogical proportion* if $I_2(x, y) \equiv I_2(z, t)$ and $I_3(x, y) \equiv I_3(z, t)$, *i.e.*, if $x \wedge \overline{y} \equiv z \wedge \overline{t}$ and $\overline{x} \wedge y \equiv \overline{z} \wedge t$ (here, $\equiv$ denotes the logical equivalence). Let $\mathbf{C}_x$, $\mathbf{C}_y$, $\mathbf{C}_z$, and $\mathbf{C}_t$ be the feature categories constructed as in Fig. 4. The category $\mathbf{C}_x$ contains the

$$\begin{tikzcd}
\overline{x} \arrow[loop left, "\overline{x} \to \overline{x}"] \arrow[r, bend right, "\overline{x} \to x"'] & x \arrow[l, bend right, "x \to \overline{x}"'] \arrow[loop right, "x \to x"]
\end{tikzcd}$$

Fig. 4: The category $\mathbf{C}_x$, with two objects (x and $\overline{x}$) for a propositional variable x, and arrows represent changes between these values.

two objects x and $\overline{x}$ for the propositional variable x. The interpretation functor $.^{\mathcal{I}}$ is defined using the valuation function v, which is a function from the set of propositional variables to $\{0, 1\}$, seen as the class of all subsets of a one-element set (0 is the empty set and 1 is the one-element set):

$$\begin{aligned}
a^{\mathcal{I}} &= v(a) \in \{0, 1\} && \text{for an object } a \text{ of } \mathbf{C}_x \\
(a \to b)^{\mathcal{I}} &= a^{\mathcal{I}} \times b^{\mathcal{I}} \subseteq \{0, 1\} \times \{0, 1\} && \text{for an arrow } a \to b \text{ of } \mathbf{C}_x
\end{aligned}$$

The arrow $(x \to \overline{x}, \overline{y} \to y)$ of the product $\mathbf{C}_x \times \mathbf{C}_y$ is interpreted as the binary relation $(x \to \overline{x}, \overline{y} \to y)^{\mathcal{I}} = v(x \wedge \overline{y}) \times v(\overline{x} \wedge y)$. Two pairs (x, y) and (z, t) are in analogical proportion if the interpretation of the two arrows $(x \to \overline{x}, \overline{y} \to y)$ and $(z \to \overline{z}, \overline{t} \to t)$ are the same, *i.e.*, if $(x \to \overline{x}, \overline{y} \to y)^{\mathcal{I}} = (z \to \overline{z}, \overline{t} \to t)^{\mathcal{I}}$. Likewise, two pairs (x, y) and (z, t) would be in paralogy if the interpretation of the arrows $(x \to \overline{x}, y \to \overline{y})$ and $(z \to \overline{z}, t \to \overline{t})$ are the same.

6.2 Similarity Paths

Let $\mathbf{C}$ be a feature category. A *similarity path* of $\mathbf{C}$ is a combination of arrows of $\mathbf{C}$. For example, $\xrightarrow{\texttt{d}\to\texttt{d}} \cdot \xrightarrow{\texttt{d}\to\texttt{m}}$ is a similarity path in the category **Area**. The free category $\mathcal{F}(\mathbf{C})$ generated by $\mathbf{C}$ is the category that has the paths of $\mathbf{C}$ as arrows. This definition can be extended to the product $\Pi = \mathbf{C}_1 \times \mathbf{C}_2 \times \ldots \times \mathbf{C}_n$ of n comparison categories $\mathbf{C}_1$, $\mathbf{C}_2$,...,$\mathbf{C}_n$. A path in Π is an arrow of the free category $\mathcal{F}(\Pi)$ generated by Π. For example, $(\xrightarrow{\mathtt{d}\to\mathtt{d}},\xrightarrow{<})\cdot(\xrightarrow{\mathtt{d}\to\mathtt{m}},\xrightarrow{=})$ is a path in the free category generated by the product $\mathbf{Area} \times \mathbf{C}_<$.

The interpretation of a similarity path on the set $\mathcal{X}$ is given by the interpretation functor $.^{\mathcal{I}}$ which by definition of functors, preserves composition: $(\xrightarrow{c}\cdot\xrightarrow{d})^{\mathcal{I}} = (\xrightarrow{d})^{\mathcal{I}} \circ (\xrightarrow{c})^{\mathcal{I}}$. Here, the composition operation $\circ$ on the arrows of the category **Rel** is the usual composition of binary relations. This definition can also be extended to the product $\Pi = \mathbf{C}_1 \times \mathbf{C}_2 \times \ldots \times \mathbf{C}_n$ of n feature categories $\mathbf{C}_1$, $\mathbf{C}_2$,...,$\mathbf{C}_n$: for two sets of arrows $c_i, d_i \in \mathbf{C}_i$,

$$((\xrightarrow{c_1},\ldots,\xrightarrow{c_n})\cdot(\xrightarrow{d_1},\ldots,\xrightarrow{d_n}))^{\mathcal{I}} = (\xrightarrow{d_1},\ldots,\xrightarrow{d_n})^{\mathcal{I}} \circ (\xrightarrow{c_1},\ldots,\xrightarrow{c_n})^{\mathcal{I}}$$

For example, for an apartment $\mathtt{srce} \in \mathcal{X}$ located in **d**owntown, and an apartment $\mathtt{tgt} \in \mathcal{X}$ located in **m**idtown, the pair $(\mathtt{srce},\mathtt{tgt})$ is in the interpretation of the similarity path $(\xrightarrow{\mathtt{d}\to\mathtt{d}},\xrightarrow{<})\cdot(\xrightarrow{\mathtt{d}\to\mathtt{m}},\xrightarrow{=})$ if there is an apartment $\mathtt{pb}$ such that $\mathtt{srce}\,(\xrightarrow{\mathtt{d}\to\mathtt{d}},\xrightarrow{<})^{\mathcal{I}}\,\mathtt{pb}\,(\xrightarrow{\mathtt{d}\to\mathtt{m}},\xrightarrow{=})^{\mathcal{I}}\mathtt{tgt}$, that is, such that the location of $\mathtt{pb}$ is **d**owntown and its price is strictly greater than the price of $\mathtt{srce}$, and equal to the price of $\mathtt{tgt}$. This definition is consistent with the notion of similarity path, which is defined in [16] as a sequence of relations

$$\mathtt{srce} = \mathtt{pb}_0\,\mathtt{r}_1\,\mathtt{pb}_1\,\mathtt{r}_2\,\mathtt{pb}_2\ldots\mathtt{pb}_{q-1}\,\mathtt{r}_q\,\mathtt{pb}_q = \mathtt{tgt}$$

such that the $\mathtt{pb}_i$'s are problems and $\mathtt{r}_i$'s are binary relations between problems.

7 Ordering Differences

7.1 A Subsumption Relation

A subsumption operator $\sqsubseteq$ enables to order comparisons by generality. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two feature categories. For an arrow $\xrightarrow{c_1}$ of $\mathbf{C}_1$, and an arrow $\xrightarrow{c_2}$ of $\mathbf{C}_2$, we write $\xrightarrow{c_1}\sqsubseteq\xrightarrow{c_2}$ to represent that whenever an A can be compared to B using the comparison $\xrightarrow{c_1}$, then A can be compared to B using comparison $\xrightarrow{c_2}$. For example, in $\Pi = \mathbf{Area} \times \mathbf{C}_<$, the subsumption relation $\xrightarrow{\mathtt{d}\to\mathtt{m}}\sqsubseteq\xrightarrow{<}$ represents the fact that any apartment located in **d**owntown is more expensive than any apartment located in **m**idtown. The subsumption operator $\sqsubseteq$ can also relate the arrows of two product categories $\Pi_C = \mathbf{C}_1 \times \mathbf{C}_2 \times \ldots \times \mathbf{C}_k$ and $\Pi_D = \mathbf{D}_1 \times \mathbf{D}_2 \times \ldots \times \mathbf{D}_\ell$. For example, if $\Pi_C = \mathbf{C}_< \times \mathbf{Area}$ represents comparisons between the number of rooms and the location of apartments, and $\Pi_D = \mathbf{C}_<$ represents comparisons in price, then $(\xrightarrow{=},\xrightarrow{\mathtt{m}\to\mathtt{d}})\sqsubseteq\xrightarrow{<}$ represents the fact that for a same number of rooms, an apartment located in **d**owntown is more expensive than an apartment located in **m**idtown.

Subsumption relations $\xrightarrow{c_1}\sqsubseteq\xrightarrow{c_2}$ are interpreted as set inclusions in $\mathcal{X} \times \mathcal{X}$:

$$\xrightarrow{c_1}\sqsubseteq\xrightarrow{c_2} \text{ if } \xrightarrow{c_1}^{\mathcal{I}} \subseteq \xrightarrow{c_2}^{\mathcal{I}}$$

This definition extends naturally to product categories:

$$(\xrightarrow{c_1},\ldots,\xrightarrow{c_k}) \sqsubseteq (\xrightarrow{d_1},\ldots,\xrightarrow{d_\ell}) \text{ if } (\xrightarrow{c_1},\ldots,\xrightarrow{c_k})^{\mathcal{I}} \subseteq (\xrightarrow{d_1},\ldots,\xrightarrow{d_\ell})^{\mathcal{I}}$$

A subsumption relation corresponds to the notion of *co-variation*, that is defined in [4] as a functional dependency between variations, and may be used to represent adaptation rules.

7.2 Analogical Jump

An analogical "jump" consists in making the hypothesis that a subsumption relation on comparisons holds for a given pair of objects. From a logical point of view, an analogical jump is defined in [7] as the following hypothetical rule of inference:

$$\text{if } P(x) = P(y) \text{ and } Q(x), \text{ then we can infer } Q(y)$$

For example, Bob's car and John's car share the property P of being a 1982 Mustang GLX V6 hatchbacks, and Bob's car has the property Q of having a price of 3500 \$. The inference is that the price of John's car should also be around 3500 \$. This schema can be rephrased using comparisons:

$$\text{from } x \xrightarrow{P} y, \text{ infer } x \xrightarrow{Q} y$$

In this schema, $\xrightarrow{P}$ and $\xrightarrow{Q}$ are two comparisons representing respectively that an element shares the property P with another element, and that it shares the property Q. This inference consists in making the hypothesis that the subsumption relation $\xrightarrow{P} \sqsubseteq \xrightarrow{Q}$ on comparisons holds for the pair (x, y). Such inference can also be made when the comparisons represent differences. For example, if $\Pi_C = \mathbf{C}_< \times \mathbf{Area}$ represents comparisons between the number of rooms and the location of apartments, and $\Pi_D = \mathbf{C}_<$ represents comparisons in price, then the subsumption relation $(\xrightarrow{=}, \xrightarrow{\mathtt{m}\to\mathtt{d}}) \sqsubseteq \xrightarrow{<}$ can be applied to a pair (x, y) of apartments to infer that an apartment y located in **d**owntown is more expensive than an apartment x with the same number of rooms, but located in **m**idtown.

8 Conclusion

Category Theory seems to be a natural setting to represent the feature comparisons made when establishing case differences. We showed that it can be used to and to design a qualitative language in which both case differences, similarity paths and adaptation rules can be represented and reasoned upon symbolically. We believe that such results open the way to new qualitative formalizations of the case-based inference, that would be able to integrate both retrieval and adaptation in a same analogical process.

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